Exercise Session 5

Lecture summary: LQG

- State at time t: $X_t \in \mathbb{R}^n$
- Action or control at time t: $U_t \in \mathbb{R}^m$
- Dynamics $X_{t+1} = \underbrace{A}_{\mathbb{R}^{n\times n}} X_t + \underbrace{B}_{\mathbb{R}^{n\times m}} U_t + \underbrace{\mathcal{E}_{t+1}}_{\mathbb{R}^n \text{ noise}}$,

- Cost: Quadratic

  cost of action u for t<T

$$C(x,u) = x^T R x + u^T Q u + x^T S^T u + u^T S x$$

$$= \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

  cost of terminal State $C(x) = x^T \Pi_T x$.

- Objective: Find a controller $u_0, \dots, u_{T-1}$ minimizing

$$\mathbb{E}\left[\sum_{t=0}^{T-1} C(X_t, U_t) + C(X_T) \right]$$

- Perfect Observations

- no noise: $X_t = A X_{t-1} + B U_{t-1}$; cost $\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} \Pi_{xx} & \Pi_{xu} \\ \Pi_{ux} & \Pi_{uu} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$

min. is achieved by $\boxed{u = -\Pi_{uu}^{-1} \Pi_{ux} \cdot x}$ and is equal $\boxed{x^T [\Pi_{xx} - \Pi_{xu} \Pi_{uu}^{-1} \Pi_{ux}] x}$

Solution by Riccati equation: for all t<T $\boxed{V_t^*(x) = x^T \Pi_t x}$

where $\Pi_t = R + A^T \Pi_{t+1} A - (S^T + A^T \Pi_{t+1} B)(Q + B^T \Pi_{t+1} B)^{-1} (S + B^T \Pi_{t+1} A)$

The optimal control is linear: $\boxed{U_t = K_t \cdot X_t}$, where

$$K_t = -(Q + B^T \Pi_{t+1} B)^{-1} (S + B^T \Pi_{t+1} A)$$

- with white noise $\boxed{V_t^*(x) = x^T \Pi_t x + \sum_{i=t+1}^{T} tr(N \Pi_i)}$ for all t

where $N = \mathbb{E}[\mathcal{E}_t \mathcal{E}_t^T]$, optimal control: $\boxed{U_t = K_t \cdot X_t}$ same as without noise

- $\boxed{\text{Imperfect observations.}}$

System dynamics : $\begin{cases} x_{t+1} = A x_t + B u_t + \varepsilon_{t+1} \\ y_{t+1} = C x_{t-} + \eta_{t+1} \end{cases}$

Observation at time t: $y_t \in \mathbb{R}^p$, $C \in \mathbb{R}^{p \times n}$

white noise: $\mathbb{E}\begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} = 0 \quad \mathbb{E}\begin{bmatrix} \varepsilon_t \varepsilon_t^T & \varepsilon_t \eta_t^T \\ \eta_t \varepsilon_t^T & \eta_t \eta_t^T \end{bmatrix} = \begin{pmatrix} N & L \\ L^T & M \end{pmatrix}$

observed history at time t: $W_t = (x_0, u_0, y_1, u_1 \cdots y_{t-1}, u_{t-1}, y_t)$

State estimate: $\hat{x}_t = \mathbb{E}[x_t \mid W_t]$

Estimation error $\Delta_t = x_t - \hat{x}_t$

- Kalman Filter: Assume that $x_0 \sim N(\hat{x}_0, V_0) \Rightarrow$ for a given history W_t, $x_t \sim N(\hat{x}_t, V_t)$, where $\hat{x}_t$ and V_t can be computed recursively.

$\boxed{\text{Kalmal Filter:}}$ $\hat{x}_t = A\hat{x}_{t-1} + B u_{t-1} + H_t(y_t - C\hat{x}_{t-1})$

$\boxed{\text{Riccati}}$: $V_t = G(V_{t-1})$

with

$$H_t = (L + A V_{t-1} C^T)(M + C V_{t-1} C^T)^{-1}$$

$$G(V) = N + AVA^T - (L + AVC^T)(M + CVC^T)^{-1}(L^T + CVA^T)$$

Ex. 4. (Exam question)

$X_0 = 1; \quad X_{t+1} = X_t + b\,U_t, \quad b > 0$

Goal: minimize $J(T) = \sum_{t=0}^{T-1} (X_t^2 + \rho\, U_t^2)$

$\boxed{\text{a) Find the optimal Control sequence}}$

We have: $A = 1,\ B = b,\ R = 1,\ Q = \rho,\ S = 0,\ \Pi_T = 0$

Riccati equation: $V_T^*(X) = 0$; $\forall t < T$, $V_t^*(x) = X^T \Pi_t X$, where $\Pi_t = (\cdots)$

$$\Rightarrow \Pi_t = 1 + \Pi_{t+1} - (b \cdot \Pi_{t+1}) \cdot (\rho + b^2 \Pi_{t+1})^{-1} \cdot (b\, \Pi_{t+1}) = 1 + \frac{\rho\, \Pi_{t+1}}{\rho + b^2 \Pi_{t+1}}$$

The optimal control: $U_t = K_t \cdot X_t$

$$K_t = -(\rho + \beta^2 \Pi_{t+1})^{-1} \cdot (\beta \cdot \Pi_{t+1})$$

$$\Rightarrow U_t = -\frac{b\, \Pi_{t+1}}{\rho + b^2 \Pi_{t+1}} \cdot X_t$$

$\boxed{\text{b) Under which condition, } X_{t+1} = a \cdot X_t;\ 0 \le a < 1 \text{ when } T \to \infty}$

When t grows, Π_t converges to the fix point $f(x) = \frac{1 + \rho x}{\rho + b^2 x}$

$$\Rightarrow \Pi_\infty = 1 + \frac{\rho\, \Pi_\infty}{\rho + b^2 \Pi_\infty}$$

$$\cancel{\rho\, \Pi_\infty} + b^2 \Pi_\infty^2 = \rho + b^2 \Pi_\infty + \cancel{\rho\, \Pi_\infty}$$

$$b^2 \Pi_\infty^2 - b^2 \Pi_\infty - \rho = 0$$

$$\Pi_\infty = \frac{b^2 \pm \sqrt{b^4 + 4 b^2 \cdot \rho}}{2 b^2} = \frac{1}{2} \cancel{\pm} \frac{1}{2}\sqrt{1 + \frac{4\rho}{b^2}} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\rho}{b^2}}$$

Hence, the asymptotic dynamic becomes:

$$X_{t+1} = X_t + b \cdot \left(-\frac{b\, \Pi_\infty}{\rho + b^2 \Pi_\infty} \cdot X_t\right) = X_t \underbrace{\left(1 - \frac{\Pi_\infty}{\frac{\rho}{b^2} + \Pi_\infty}\right)}_{<1}$$

c) $X_{t+1} = X_t + b u_t + \xi_{t+1}$ $\quad y_t = x_t + \eta_t$

Using the lecture notes notation

$$N = 1, \quad L = \beta, \quad M = \alpha$$

Kalman Filter: $\hat{X}_t = \hat{X}_{t-1} + b u_{t-1} + H_t \cdot (y_t - \hat{X}_{t-1})$

$$H_t = (\beta + V_{t-1})(\alpha^2 + V_{t-1})^{-1}$$

Ricatti recursion: $V_t = 1 + V_{t-1} - \dfrac{(\beta + V_{t-1})^2}{\alpha^2 + V_{t-1}}$

when $t \to \infty$ V_t converges to V_∞ =>

$$V_\infty = 1 + V_\infty - \frac{(\beta + V_\infty)^2}{\alpha^2 + V_\infty}$$

Solve for V_∞ :

$$V_\infty = \frac{1}{2}\left(1 - 2\beta + \sqrt{\alpha^2 + (2\beta - 1)^2}\right)$$

We can show that $\dfrac{\partial V_\infty}{\partial \beta} < 0$ => variance is smaller when β is increasing

=> positive β is better.

Ex. 1.

Assume $\tilde{X}$ is the state trajectory for $X_0 = 0 \Rightarrow$

$$\left.\begin{aligned} X_1 &= A\cdot z + Bu_0 \\ \tilde{X}_1 &= \qquad\; Bu_0 \end{aligned}\right\} \Rightarrow X_1 = \tilde{X}_1 + Az$$

$$\left.\begin{aligned} X_2 &= AX_1 + Bu_1 = A(Az + Bu_0) + Bu_1 \\ \tilde{X}_2 &= A\tilde{X}_1 + Bu_1 = A(Bu_0) + Bu_1 \end{aligned}\right\} X_2 = A^2\cdot z + \tilde{X}_2$$

$\vdots$

we can show that $X_t = A^t\cdot z + \tilde{X}_0$ for the given $X_0 = z$

Then, The total cost:

$$J = \underbrace{\sum_{t=0}^{\infty} \lambda^t\cdot g^T A^t\cdot X_0}_{\displaystyle \| \atop \displaystyle g^T \sum_{t=0}^{\infty} (\lambda A)^t\cdot X_0 = g^T(I - \lambda A)^{-1}\cdot X_0} + \sum (\lambda^t g^T\cdot\tilde{X}_t + \lambda^t u_t^T R u_t)$$

Next, we show that $V(z)$ is affine

$V(z) = \inf_u J$, we estimate V at z_1 and z_2

$$V(z_1) = \inf_u \left(g^T(I - \lambda A)^{-1}\cdot z_1 + \sum_{t=0}^{\infty} \lambda^t \left(g^T \tilde{X}_t + u_t^T R u \right)\right) \qquad (1)$$

$$V(z_2) = \inf_u \left(g^T(I - \lambda A)^{-1}\cdot z_2 + \sum_{t=0}^{\infty} \lambda^t \left(g^T \tilde{X}_t + u_t^T R u \right)\right) \qquad (2)$$

$(2)-(1) \Rightarrow V(z_2) - V(z_1) = \underbrace{g^T(I - \lambda A)^{-1}}\cdot(z_2 - z_1) \Rightarrow$

$V(z)$ is affine, i.e. $V(z) = p^T z + \boxed{C}$?? , where $\boxed{p^T = g^T(I - \lambda A)^{-1}}$

On the other hand, using Bellman equation we get

$$V(x) = \min_u \left(g^T x + u^T R u + \lambda\cdot V(Ax + Bu)\right)$$

$$p^T x + C = \min_u \left(g^T x + u^T R u + \lambda\cdot p^T(Ax + Bu) + \lambda C\right)$$

We minimize the right hand side by setting the derivative w/ resp. to u to 0.

Then: $\quad 2Ru + \lambda B^T p = 0 \Rightarrow u^* = -\frac{1}{2}\lambda R^{-1} B^T p$

$\Rightarrow$

$$p^T x + c = g^T x + \frac{\lambda^2}{4} p^T B R^{-1} R R^{-1} B^T p + \lambda p^T \left(Ax + B\left(-\frac{1}{2}\lambda R^{-1} B^T p\right)\right) + \lambda \cdot c$$

$$c(1-\lambda) = g^T x - p^T x - \frac{\lambda^2}{4} \cdot p^T B R^{-1} B^T p + \lambda p^T A x$$

$$c = \frac{g^T x - p^T(x - \lambda A x) - \frac{\lambda^2}{4} p^T B R^{-1} B^T p}{1-\lambda}$$

$$c = \frac{\cancel{g^T x - g^T (1-\lambda A)^{-1} (1-\lambda A) x} - \frac{\lambda^2}{4} p^T B R^{-1} B^T p}{(1-\lambda)}$$

$$c = -\frac{\lambda^2}{4(1-\lambda)} \cdot p^T B R^{-1} B^T p$$

Plugging back into V we get.

$$\boxed{V(x) = p^T x - \frac{\lambda p^T B R^{-1} B^T p}{4(1-\lambda)}} \quad \text{where } p = (I - \lambda A)^{-1} \cdot g$$